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Surfaces Invariant under Infinite Discontinuous Birational Groups Defined by Line Congruences.

BY VIRGIL SNYDER.

A family of quadric surfaces containing one variable parameter will define a congruence of lines having the envelope for focal surface. When the parameter enters rationally to the power n , the congruence will be of order $2n$ and of class $6n$; the focal surface will be of order $4(n-1)$. Each quadric will touch it along a space quartic of the first kind, defined as the limiting intersection of two quadrics which approach coincidence.

1. When the two systems of generators are rationally separable, the congruence breaks up into two, of the form $(n, 3n)$, having the same focal surface. These surfaces are interesting from the fact that the congruences will define upon them an infinite discontinuous group of birational transformations which leave the surface invariant.*

Let

$$\frac{\sum_{i=0}^m a_i \lambda^{m-i} \mu^i}{\sum_{k=0}^n b_k \lambda^{n-k} \mu^k} = \frac{\sum_{i=0}^m c_i \lambda^{m-i} \mu^i}{\sum_{k=0}^n d_k \lambda^{n-k} \mu^k} = \sigma, \quad (1)$$

a_i, b_k, c_i, d_k being linear expressions in four homogeneous variables (x) , be the equations of a line. As $\lambda:\mu$ and σ take various values, this line will define a congruence which we shall call the σ congruence. Similarly, by interchanging $\Sigma c, \Sigma b$, a new congruence τ is defined.

These two congruences are arranged on a system of quadrics, and have the same focal surface ϕ . The cones contained in this family evidently belong to both congruences, and their vertices are singular points on ϕ . Should a cone further break up into a pair of planes π_1, π_2 , the corresponding lines of σ would

* Cf. "Infinite Discontinuous Groups of Birational Transformations Which Leave Certain Surfaces Invariant," *Transactions American Math. Society*, Vol. XI (1910), pp. 15-24.

belong to two pencils $(0_1, \pi_1)$, $(0_2, \pi_2)$, and the lines of τ to the pencils $(0_1, \pi_2)$, $(0_2, \pi_1)$, wherein $0_1, 0_2$ are two points on the line of intersection of the two planes. The points $0_1, 0_2$ are singular points on ϕ , and π_1, π_2 are double planes, touching ϕ along conics.

Through any point $P_1 \equiv (x_1)$ on ϕ pass a line of σ and a line of τ which touch the surface at that point.

2. When the coordinates of P_1 are substituted in (1), the binary quartic in $\lambda:\mu$ will have a double root which can be expressed rationally in terms of (x_1) . The corresponding $\lambda_1:\mu_1$ determines the single quadric $Q(\lambda_1)$ which touches ϕ at P_1 . The curve of contact c_4 is defined by $Q(\lambda_1) = 0, \frac{\partial Q}{\partial \lambda_1} = 0$. A line of σ touching ϕ at P_1 will therefore touch it again at the second point of intersection with $\frac{\partial Q}{\partial \lambda_1}$. Since the coordinates of one point of intersection are given, those of the other point $P_2 = (x_2)$ are expressed rationally in terms of (x_1) . Had we started with P_2 , we could express the coordinates of P_1 rationally in terms of (x_2) .

The operation of interchanging the points of contact of any line of σ with the focal surface ϕ is a birational transformation of order 2 which leaves ϕ invariant.

This property is true for a large category of congruences that cannot be expressed in the form (1), but is not true for those defined by a family of quadrics in which the two systems of generators are not rationally separable, for, corresponding to a point P_1 we have two points P_2 , one on each generator of the quadric passing through P_1 .*

In the same way, the congruence τ will define a birational transformation of order 2. Let these be denoted by $(\Sigma), (T)$.

3. The coordinates of a point on c_4 can be expressed in terms of elliptic functions in such a way that the parameters u of four points lying in a plane will have a sum conjugate to zero. A quadric can be passed through any c_4 and one bisecant. Let u_1, u_2 be two points such that $u_1 + u_2 = k$ is not commen-

* A congruence of this kind can be obtained from the cubic variety Γ in space of four dimensions, which contains a plane, but cannot be generated by trilinear systems. Let a pencil of spaces have the plane for basis. Each space will cut Γ in a residual quadric, one passing through each point. If the quadrics be projected from a point on Γ into any space not passing through the center of projection, the result is a family of index 2, $\lambda^2 u + \lambda \mu v + \mu^2 w = 0$. The apparent contour is the envelope of this system, and complete focal surface of the congruence. The surface $4uw - v^2 = 0$ has eight double points, and is of class 20. The vertices of the cones contained in the family of quadrics are not in this case double points of the focal surface.

surable with the periods of the elliptic functions concerned. The planes passed through this line will contain a generator of the other system such that the sum of the two parameters will be conjugate to $-k$. In particular, if u_1 define P_1 , then P_2 is expressed by $k - u_1$; from P_2 a generator of τ meets c_4 again in $u_1 - 2k$. If the operation (ΣT) be repeated, the point P_1 can never be reached after a finite number of transformations, since k was chosen incommensurable with the periods. If the constants be so chosen for one value of λ , it insures that (ΣT) is of infinite order.

4. The order of the discriminant is in general $2(m + n - 1)$, but will contain as factor any letter common to the coefficient of λ^{m+n} , $\lambda\mu^{m+n-1}$, and similarly for $\lambda\mu^{m+n-1}$, μ^{m+n} . Thus the order of ϕ is in general of order a multiple of 4, but may be reduced by one unit if $c_0 = 0$, $b_0 = 0$, $a_1 = a_0$; by two units if $d_1 = a_1$; and by three units if also $b_n = c_m = d_n = 0$; hence:

Surfaces ϕ of any order except 5 can be constructed on which the operation (ΣT) defines an infinite discontinuous group of birational transformations under which ϕ remains invariant.

5. Among the congruences expressible in the form (1) are the Cremona congruences, obtained by connecting corresponding points of two planes in (1, 1) correspondence.*

The general theory will not be discussed further, but a few interesting applications will be made to well-known congruences.

Congruences of Order 2.

6. Given the three trilinear systems in space of four dimensions S_4 ,

$$\alpha u_1 + \beta u_2 + \gamma u_3 = 0, \quad \alpha v_1 + \beta v_2 + \gamma v_3 = 0, \quad \alpha w_1 + \beta w_2 + \gamma w_3 = 0.$$

They will generate the cubic variety Γ defined by

$$\Gamma \equiv \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

Without restricting the system, we may put:

$$\begin{aligned} u_1 &= x_1, & v_1 &= x_4, & w_1 &= b_1 x_1 + b_2 x_2 + b_3 x_3 = b, \\ u_2 &= x_2, & v_2 &= x_5, & w_2 &= c_1 x_1 + c_2 x_2 + c_4 x_4 = c, \\ u_3 &= x_3, & v_3 &= a_1 x_1 + a_2 x_2 + a_3 x_3 = a, & w_3 &= d_1 x_1 + d_2 x_2 + d_5 x_5 = d. \end{aligned}$$

* T. A. Hirst: "On Cremonian Congruences," *Proc. London Math. Society*, Vol. XIV (1883), pp. 259-301. The treatment is synthetic, and no transformations are discussed.

If now a general line of the system be projected from $(0, 0, 0, 0, 1)$ into the space $x_5 = 0$, its equations become

$$\alpha x_1 + \beta x_2 + \gamma x_3 = 0, \quad d_5 \gamma (\alpha x_4 + a \gamma) = b \alpha \beta + \beta^2 c + \beta \gamma d.$$

By eliminating α the equation may be written

$$(b x_2 - c x_1) \beta^2 - (d_5 x_2 x_4 - b x_3 + d x_1) \beta \gamma + (a x_1 - x_3 x_4) d_5 \gamma^2 = 0, \quad (2)$$

showing that the congruence can be arranged in a family of quadrics of index 2. The equation of the focal surface may be written in the form

$$\phi_4 = (d_5 x_2 x_4 + b x_3)^2 + x_1 [2 d d_5 x_2 x_4 + d^2 x_1 - 2 d b x_3 - 4 c x_3 x_4 - 4 a b d_5 x_2] = 0.$$

The two systems of generators on the quadric (2) are rationally separable, and may be written

$$\frac{b \beta - d_5 x_4 \gamma}{x_1} = \frac{c \beta^2 + d \beta \gamma - a d_5 \gamma^2}{\beta x_2 + \gamma x_3} = \sigma,$$

$$\frac{b \beta - d_5 x_4 \gamma}{c \beta^2 + d \beta \gamma - a d_5 \gamma^2} = \frac{x_1}{\beta x_2 + \gamma x_3} = \tau.$$

Since the congruences are each of class 6, and the focal surface has a double plane $x_1 = 0$, they are of symbol $(2, 6)_{II}$. The double points of the focal surface consist of the eight points common to the quadrics

$$b x_2 - c x_1 = 0, \quad d_5 x_2 x_4 - b x_3 + d x_1 = 0, \quad a x_1 - x_3 x_4 = 0,$$

and the vertices of the four quadric cones of the family. Of the former, two have an exceptional function: one is the vertex of a quintic cone K_5 having a double edge passing through six of the basis points, and the other is the vertex of a plane pencil K_1 lying in the singular plane. The other six basis points are vertices of cubic cones. The vertices of the quadric cones contained in the family of quadrics are all in the singular plane $x_1 = 0$.

In σ, τ the two cones K_5, K_1 are interchanged, each has a cubic cone at the other six basis points, but the cones are not the same for both. The quadric cones are common to σ and τ . The equations of these cones can be determined by finding the envelope of the plane passing through the singular point. One equation of the line of the congruence is illusory, and the parameters in the other will satisfy a relation with constant coefficients.

7. When the operation (Σ) is performed upon the focal surface, all the double points are principal elements. K_1 will go into the conic of contact lying in $x_1 = 0$, each of the K_2 vertices goes into a space quartic having a double point at

the vertex, the K_3 each into a sextic having a triple point and K_6 into a curve of order 10, with a five-fold point at the vertex, and passing through the basis points. All these curves are rational.

8. Given any plane section π of ϕ . Consider the ruled surface formed by all the lines of σ which touch ϕ along the quartic curve of section. Its order will be 4 plus the number of generators lying in π . Since the class of σ is 6, there are six bitangents lying in π , each counting as a double generator. Thus the order of the ruled surface is 16. Since its genus is 3, the residual double curve is of order 96, no part of which can consist of further double generators. The plane π cuts the curve c_{2i} , image of K_i , in $2i$ points, hence $2i$ generators of our ruled surface pass through K_i . The surfaces ϕ_4 and R_{16} intersect in a curve of order 64, consisting of c_4 in π counted twice, and a residual c_{28} , also counted twice. The latter will have $2i$ branches passing through every K_i . The double curve D_{96} on R_{16} meets ϕ_4 in all the vertices K_i and nowhere else. The plane π will cut each cone K_i in a plane c_i which touches c_4 in $2i$ points. The lines joining these points to the vertex K_i will therefore be torsal generators, the pinch-point of each being at the vertex. The total intersection of D_{96} and ϕ_4 is therefore of the form

$$2 \left\{ \frac{10 \cdot 9}{2} + 5 + 6 \left(\frac{6 \cdot 5}{2} + 3 \right) + 4 \left(\frac{4 \cdot 3}{2} + 2 \right) + 1 + 1 \right\}.$$

9. If between the equations of a line of the congruence the parameter β be eliminated, a family of ruled cubics of index 2 results,

$$\alpha^2 [b x_1 x_2 - c x_1^2] + \alpha \gamma [d_5 x_2^2 x_4 + b x_2 x_3 - 2 c x_1 x_3 + d x_1 x_2] + \gamma^2 [d_5 a x_2^2 - c x_3^2 + d x_2 x_3] = 0,$$

which has ϕ for its envelope, with the extraneous factor x_2^2 . The double directrices are defined by

$$x_2 = 0, \quad \alpha x_1 + \beta x_3 = 0.$$

In this way we can obtain the properties first found by Stahl.*

10. The equations

$$\frac{a \lambda + b \mu}{c \lambda + d \mu} = \frac{x \lambda + y \mu}{z \lambda + w \mu} = \sigma$$

and the corresponding ones for τ also define congruences of order 2 and class 6, but the focal surface has no double plane; hence the congruence is of the first kind $(2, 6)_I$. The configuration of singular points and fundamental curves can

* W. Stahl: "Ueber Strahlensysteme zweiter Ordnung," *Crelle's Journal*, Vol. XCV (1883), pp. 297-316.

be found as before. The essential difference between the two forms is that in $(2, 6)_{II}$ the vertices of the quadric cones belonging to the family of quadrics lie in a plane, while in $(2, 6)_I$ they do not. The present form can not be obtained from the cubic variety Γ of S_4 .

11. If in the equation Γ the space v_3 is of the form $a_4x_4 + a_5x_5$, the entire plane $x_4 = 0$, x_5 lies on Γ . There is no restriction in putting $a = 0$, since the equation of any space through this plane is of the form $kx_4 + lx_5 = 0$. Each space of this pencil will cut Γ in a quadric whose generators belong to conjugate systems. Four double points of Γ lie in the plane, and are basis points of the pencil of conics cut from the quadrics by the plane $x_4 = 0$, $x_5 = 0$.

On eliminating α as before, the system

$$\frac{b\beta - d_5x_4\gamma}{x_1\beta} = \frac{c\beta + d\gamma}{x_2\beta + x_3\gamma} = \sigma$$

is seen to be a particular case of both the preceding forms. The planes $x_1 = 0$, $x_3 = 0$ and $x_4 = 0$ are double planes of the focal surface.

From the equation of the system of quadrics

$$\beta^2(bx_3 - cx_1) + \beta\gamma(bx_3 - d_5x_3x_4 - dx_1) - \gamma^2d_5x_3x_4 = 0$$

it is seen that four of the eight basis points lie in the plane $x_3 = 0$, and the other four lie in $x_4 = 0$. Besides the four cones whose vertices are in $x_1 = 0$, the congruence has one other singular point. This is the general congruence $(2, 5)$.

12. If $b = 0$, the general congruence $(2, 4)$ results. The singular planes of the focal surface are $x_1 = 0$, $x_3 = 0$, $x_4 = 0$, $c = 0$.

Finally, the $(2, 3)$ and $(2, 2)$ congruences can be obtained from the equations given in a preceding paper.*

In the case of the general $(2, 2)$ congruence, the same focal surface exists for six families of congruences. The operations $(\Sigma)(T)$ are now all commutative, and generate a group of order 32. In all the other cases enough undefined constants exist to make (ΣT) of infinite order, but for certain values of these constants they may be of finite order.

Every congruence of order 2 not having focal curves and of class less than 7 can be arranged on a family of quadric surfaces of index 2, whose envelope is the focal surface of order 4. For unrestricted coefficients these surfaces all have an infinite group of birational transformations except that of class 4.

* Snyder: "Surfaces Derived from the Cubic Variety Having Nine Double Points in Four-Dimensional Space," *Trans. Am. Math. Soc.*, Vol. X (1909), pp. 71-78.

Congruences of Order 3.

13. If the variety Γ which has no plane lying upon it be projected from a general point of S_4 , the resulting congruences of order 3 can not in general be arranged on quadrics. The two conjugate congruences of order 3 and class 6 have the same focal surface of order 6.

If, however, $a = 0$, and the system be projected from (00101) into $x_5 = 0$, an interesting case results. It will be more convenient to use a non-homogeneous parameter k . The projection of the first system gives

$$\frac{d_5 x_4 + k d}{x_4 + k x_3} = \frac{k(b - k c) + b_3 x_4}{k(x_1 - k x_2)} = \sigma \quad C_3,$$

from which

$$(c x_3 - d x_2) k^3 + (d x_1 - d_5 x_2 x_4 + c x_4 - b x_3) k^2 + (d_5 x_1 x_4 - b x_4 - b_3 x_3 x_4) k - b_3 x_4^2 = 0.$$

In the k -discriminant, x_4^2 is a factor.

Of the six quadric cones in this family of quadrics, three have their vertices in $x_4 = 0$, three in $d_5 x_3 - d = 0$. The point $(d_2, -d_1, 0, 0)$ is the vertex of a quartic cone having three cuspidal edges, belonging to C_3 .

A second system is

$$\frac{k d + d_5 x_4}{k(b - k c) + b_3 x_4} = \frac{k x_3 + x_4}{k(x_1 - k x_2)} = \tau \quad C_1.$$

This one has the same three quadric cones with vertices in the plane $x_4 = 0$, cubic cones with vertices at the singular points in the plane $d_5 x_3 - d = 0$, and a plane pencil in this plane with vertex at $(d_2, -d_1, 0, 0)$.

Finally, a third system is of the form

$$\frac{d - k x_3}{(d_5 - k)(k x_2 - c)} = \frac{x_4(d_5 - k)}{b_3(d - k x_3) + (d_5 - k)(k x_1 - b)} = \rho \quad C_2,$$

which has a pencil in the plane $x_4 = 0$. It is like C_1 , if the planes $x_4, d_5 x_3 - d = 0$ be interchanged. The other system of generators on the quadrics of C_2 is exactly C_3 . Each of the operations (C_i) is of order 2, and the product of any two is of infinite order.

To obtain the equation of the cone of rays passing through the point $(d_2, -d_1, 0, 0)$ and belonging to C_1 , substitute the coordinates of this point in C_1 and solve for τ, k . The first gives $\tau = 0$; hence $k d + d_5 x_4 = 0, k x_3 + x_4 = 0$. The k -eliminant is $(d_5 x_3 - d) x_4 = 0$. The factor x_4 corresponds to $k = 0$; hence the pencil $d_5 x_3 - d = 0, (d_2, -d_1, 0, 0)$ belongs to C_1 .

Similarly, in C_2 we have $\rho = 0$, $x_4 = 0$ for the same point, and for C_3 the coordinates of the point satisfy one equation identically. The other gives $\sigma(d_2 + k d_1) = \bar{b}$, $\bar{b}$ being $d_2 b_1 - d_1 b_2$. On substituting this value of σ in the second equation we have the equation of a plane always passing through the singular point. The envelope of this plane is a quartic cone having three cuspidal edges. In the same way the cones belonging to the other singular points are determined.

The duals of all the congruences of order 2 discussed before can also be expressed in this way.

14. Another case of considerable interest was mentioned by Roccella.* Let $(0_i, \pi_i)$ ($i = 1, 2, 3$) represent three projective plane pencils. Lines joining corresponding lines will generate a quadric, and the quadrics thus defined will compose a congruence G . The directrices of the quadrics define another congruence D . The planes π_i, π_k will intersect in a line (π_i, π_k) which is cut by the two pencils in projective ranges. Let P_{ik}, P'_{ik} be the self-corresponding points, and $d_i, d_k; d'_i, d'_k$ the pairs of lines of the two pencils passing through them. Let δ_m be the line of $(0_m, \pi_m)$ corresponding to d_i . The lines of the pencils $(P_{ik}, \delta_m), (P'_{ik}, \delta'_m)$ obviously belong to G . Let the plane (d_i, d_k) cut δ_m in D_m . The pencil $(D_m, d_i d_k)$ will also belong to G . The lines of the pencils $(P_{ik}, d_i d_k)$ and $(D_m (P_{ik}, \delta_m))$ likewise belong to D .

The quadric cones belonging to the family of quadrics therefore break up into planes. The congruences G, D have twelve singular points, at each of which the singular lines form a plane pencil.

The pencils $(0_i, \pi_i)$ belong to D , but not to G . If l_i, l_k, l_m form any triad of corresponding lines, every line through 0_i cutting l_k, l_m will evidently belong to D . The lines which meet l_k, l_m describe a quadratic complex. The complex cone having 0_i as vertex will therefore belong to G . Similarly, all the lines of π_i belonging to this complex will belong to G . From their method of generation both congruences are self-dual; hence:

The congruence generated by the lines which meet corresponding lines of three projective pencils is of order and class 3; it contains fifteen singular points and fifteen singular planes. At twelve of these points the singular lines form plane pencils, and at the others, quadric cones. The three remaining planes contain conics passing

* "Sagli enti geometrici dello spazio di rette generati dalle intersezioni de' complessi corrispondenti in due o più fasci proiettivi di complessi lineari." Piazza Armerina, 1882. See Art. 10.

through the vertices of the cones. The conjugate congruence is of order and class 3, and contains fifteen singular points, at each of which the singular lines form a plane pencil.

15. Analytically we may write

$$\begin{aligned} x_1 + \lambda x_2 &= 0, & x_3 &= 0 & \text{for } (0_1, \pi_1), \\ \lambda x_1 + x_2 &= 0, & x_4 &= 0 & \text{for } (0_2, \pi_2), \\ a_2 x_2 + a_3 x_3 + a_4 x_4 &= 0, & x_1 + x_3 + x_4 &= 0 & \text{for } (0_3, \pi_3), \end{aligned}$$

wherein $a_i = b_i + \lambda c_i$, and b_i, c_i are constants.

The equations of a line of G now have the form

$$\frac{(\lambda^2 - 1)(a_3 - a_4)x_3 - (a_2\lambda + a_4)(x_1 + \lambda x_2)}{a_2(x_1 + \lambda x_2 + x_3)} = \frac{\lambda x_1 + x_2}{x_4} = \sigma.$$

The focal surface is the same for G and D . It is of order and class 8 and has a cuspidal curve of order 12. In general the cycle (GD) is of infinite order. The surface is the self-dual surface of Kummer and Cayley.*

CORNELL UNIVERSITY, July, 1909.

* E. E. Kummer: "Ueber diejenigen Flächen, welche mit ihren reciprok polaren Flächen von derselben Ordnung sind und die gleichen Singularitäten besitzen," *Berliner Monatsberichte*, 1878, pp. 25-37; Cayley: "On Sibi-Reciprocal Surfaces," same, pp. 309-313.